A Primer on Work-Energy Relationships for Introductory Physics

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There has been and continues to be considerable discussion in the educational community about different ways of relating the concepts of work and energy in introductory physics.[1] The present article reviews a consistent and streamlined treatment of the subject, drawing particular attention to aspects seldom covered in textbooks. The paper is intended to clarify the central equations for introductory courses and to put the wider literature in context. It is specifically designed to tie closely in terminology and order of presentation to standard texts, so that it complements rather than supplants them. In brief, the key point is that there are two major categories of work, center-of-mass work and particle work.[2] After an overview of these two approaches, I illustrate them with a couple of instructive examples that can be used in group problem-solving sessions in class.

Center-of-Mass Work

In what is usually called the work-energy theorem, one is concerned with *center-of-mass* work and *mechanical* energy. This relation is most useful in mechanics, and it is a theorem in that it can be *derived* starting from Newton's laws.[3] Suppose an object[4] i has mass m_i and that net force $\mathbf{F}_i$ acts on it while its center of mass (c.m.) undergoes a differential displacement $d\mathbf{r}_i$, so that its c.m. velocity is $v_i \equiv d\mathbf{r}_i/dt$. If we take our system to be composed of a set of objects, which will be referred to from now on as *parts*, it is easy to prove that

$$W_{\text{c.m.}} = \Delta K \tag{1}$$

in an inertial reference frame, where the center-of-mass work done on the system is

$$W_{\text{c.m.}} \equiv \sum_{\text{parts}} \int \mathbf{F}_i \cdot d\mathbf{r}_i , \tag{2}$$

and the translational kinetic energy of the system is

$$K \equiv \sum_{\text{parts}} \tfrac{1}{2} m_i v_i^2 . \tag{3}$$

Note that Eq. (1) is perfectly general. In particular, it is applicable to deformable objects such as a vertical chain falling into a pile on a surface and to open systems undergoing irreversible processes such as a block sliding on a rough table.[5]

Several other comments on these equations help clarify them further:

(i) The work on individual part i is due both to external forces (i.e., exerted by agents not included in the system) and to internal forces (e.g., the force part j exerts on part i where $i \neq j$). That is, the concept of *internal work* is well defined and useful in mechanics, unlike in thermodynamics. For example, consider a system of two parts, a book and a rough table. After being given an initial push, the book comes to rest because of the internal force of friction. To the extent that the table is heavy enough that we can neglect its recoil, (negative) center-of-mass work is done only on the book and the system loses kinetic energy.

(ii) The kinetic energy of a system depends on how

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you partition it and thus one must clearly specify not only the *system* but also its *parts*. For example, consider a Frisbee® of mass m and moment of inertia I at temperature T thrown in the conventional way so that it spins with rotational speed ω as it sails through the air with translational speed v. If we take our system to be a single part, the Frisbee as a whole, then $K = \frac{1}{2}mv^2$; one might call this the *macroscopic* view. On the other hand, if we take this system to be composed of a large number of bits of plastic that are small compared to the size of the Frisbee, but large compared to molecular dimensions, then $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, which one might call the *mesoscopic* view. Finally, if one resolves the Frisbee into its N individual atoms, then $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{3}{2}Nk_BT$ (at high temperatures, typically valid at room temperature[6]), where k_B is the Boltzmann constant; this is the *microscopic* view. While this is often confusing initially to students, an appreciation of the fact that kinetic energy can be "hidden" inside an object in this way is crucial to the development of the concept of *internal energy*. To emphasize this, Eq. (2) has sometimes been called *pseudowork* in the literature,[7] although I recommend against use of this name because it is not used in standard textbooks, and this form of work is no more a "false" construct than is particle work, described in the next section.

(iii) The proof of Eq. (1) from a spatial integration of Newton's second law exactly parallels the derivation of the impulse-momentum theorem starting from a temporal integration. However, linear momentum of a system is always equal to the vector sum of the momenta of the particles composing it. Consequently one does not need to qualify impulse with adjectives such as "center-of-mass."

The rotational analog of Eq. (1) for any individual part rotating about a principal axis is

$$W_{rot} = \Delta K_{rot}, \tag{4}$$

where

$$W_{rot} \equiv \int \boldsymbol{\tau} \cdot d\theta \tag{5}$$

is the integral of the net torque on the part over its macroscopic angular displacement, and

$$K_{rot} \equiv \frac{1}{2} I\omega^2 \tag{6}$$

is the rotational kinetic energy of the part. The torque $\boldsymbol{\tau}$, differential angular displacement $d\theta$, angular speed $\boldsymbol{\omega}$, and the moment of inertia I must all be evaluated about the same axis, which must either pass through the c.m. or be the instantaneous axis of rotation, to avoid noninertial corrections. Furthermore, Eq. (4) applies only to an object whose moment of inertia is constant. As a counterexample, in the familiar demo where a point mass is swung in a circle at the end of a string of decreasing length, the rotational kinetic energy of the mass (about the center of the circle) increases even though the torque on it is always zero. Equation (4) does not apply in this case because I decreases as the string is shortened.

Equation (1) can be recast into another common form by introducing potential energy. Split the net center-of-mass work into the sum of the work done by all conservative forces, W_c, and the work done by all nonconservative forces, W_{nc}. Assuming that the system is chosen to be encompassing enough that all conservative forces are internal, W_c can be moved to the right-hand side to obtain

$$W_{nc} = \Delta E_{mech}. \tag{7}$$

Here E_{mech} is the sum of the kinetic energies K of every part and the potential energies U associated with every conservative force acting between all pairs of parts in the system.

Particle Work

At the particle[8] level, the energy of a system changes only if work is done by *external* forces,[9]

$$W_{particle} = \Delta E, \tag{8}$$

as measured by an inertial observer. Here $W_{particle}$ (sometimes called the external[1] or real[7] work) is the sum of the work done on every particle in the system by all external forces,

$$W_{particle} \equiv \sum_{particles} \int \mathbf{F}_{ext_i} \cdot d\mathbf{r}_i, \tag{9}$$

where $\mathbf{F}_{\text{ext}_i}$ is the net external force (i.e., exerted by agents external to the system) on particle i during its displacement $d\mathbf{r}_i$. (I assume that all relevant objects that do work on each other via "action at a distance" forces are included in the system to avoid the issue of the work done on and by fields. Also, for notational consistency I continue to assume that all objects that do work via conservative forces are included in the system. Both of these assumptions can be relaxed in subsequent, more advanced treatments.) The sum of the mechanical E_{mech} and internal E_{int} energies is the total energy E of the system. Internal energy[10] of a system is a sum over that of its parts. In turn, internal energy of a part is an inertial-frame-invariant state property and includes all stored energy[11] except bulk translational kinetic energy of the part's c.m. (which depends on the frame of reference of the external observer and thus is not a property of a part alone) and bulk potential energy between the parts (which depends on the interactions between them and hence is not "owned" by either part).

Equivalently one can think of particle work as a sum of the line integral of each external force over the displacement of the point of application of that force,

$$W_{\text{particle}} = \sum_{\text{forces}} \int \mathbf{F}_{\text{ext}_i} \cdot d\mathbf{r}_i \,. \tag{10}$$

For the special case where the system is isolated, so that no external work is done on it, then

$$\Delta E_{\text{mech}} = -\Delta E_{\text{int}}\,, \tag{11}$$

which is a general statement of conservation of energy. By way of examples, the mechanical energy lost by a block sliding on a rough table reappears mainly as vibrational energy of the molecules on the contacting surfaces of the table and block, and the mechanical energy gained by an accelerating figure skater comes from the chemical energy of previously eaten food.

In thermodynamics, the particle work is often categorized according to whether the energy transfer is adiabatic or thermal (i.e., driven by a temperature difference),

$$W_{\text{particle}} = W_{\text{thermo}} + Q. \tag{12}$$

This distinction[12] is unambiguous for reversible processes, as illustrated by myriad examples involving ideal gases in introductory texts. But there exist differences of opinion among educators about the magnitude of the heat transfer Q in many irreversible processes.[13] For example, during the sliding of a block on a rough table, the particles on the contacting surfaces of the block and table are neither in thermal nor mechanical equilibrium.[14] It is probably best in such situations to discuss the energy transfer between them in terms of particle work using a model such as that of Sherwood and Bernard,[15] without attempting to distinguish thermodynamic work from heat. Since the process is irreversible, Q is not directly related to the entropy change of the system in any case.

Problem 1: A pulled spool that rolls without slipping

A free-body diagram for the horizontal forces on a cylindrically symmetric spool of mass m, outer radius R, and moment of inertia I is sketched in Fig. 1.

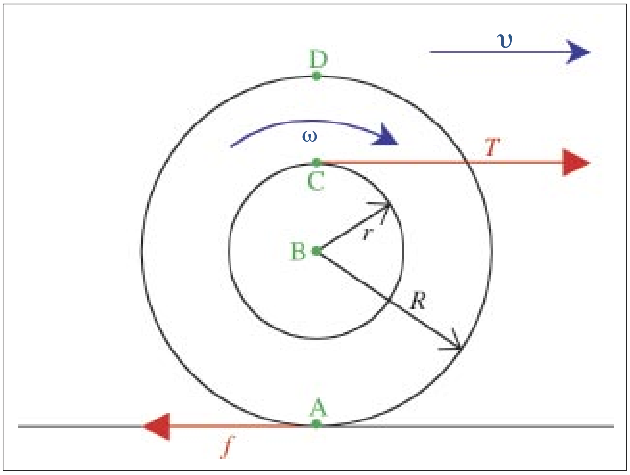

Fig. 1. Free-body diagram of a spool (with c.m. at B) being pulled by a constant tension *T* applied to its string unwrapping from the top point C of the inner cylinder of radius *r*. The spool makes contact with a rough, horizontal table at point A on its outer cylinder of radius *R*. Since the spool rolls without slipping, the friction *f* is static and is initially assumed to point opposite to the direction of pulling.

The spool starts from rest and the string (wrapped around inner radius r) is pulled horizontally with a constant tension T, causing the spool to roll without slipping a distance L along a level, rough table. Find its final translational speed v.

First consider the solution using the concept of particle work. Taking the system to be the spool (viewed macroscopically), its change in mechanical energy is $\frac{1}{2}mv^2$, while the only form of internal energy that changes is the spool's rotational kinetic energy $\frac{1}{2}I\omega^2$, where its angular speed is ω. There are two external forces, the tension T and the static friction f. The displacements of their points of application are $L(1 + r/R)$ and zero, respectively.[16] Therefore, Eq. (8) becomes

$$TL\left(1+\frac{r}{R}\right)=\tfrac{1}{2}mv^2+\tfrac{1}{2}I\omega^2. \tag{13}$$

Introducing the mass distribution factor, $\gamma \equiv I/(mR^2)$, and the no-slip condition, $\omega = v/R$, now leads to the solution,

$$v=\sqrt{\frac{2TL(1+r/R)}{m(1+\gamma)}}\ . \tag{14}$$

This solution becomes that of an object that rolls without slipping down an incline whose vertical drop is h, when we replace the external force T by mg and its distance of application $L(1 + r/R)$ by h. (Another interesting variation with the latter solution is to suspend the spool vertically by its string and allow it to fall a distance h as it freely unwinds like a yo-yo.)

Instructors need to be cognizant of a number of potential stumbling blocks for students attempting to internalize this approach:

(i) The displacements of the points of application of the external forces are not intuitively obvious. (In contrast, calculation of the center-of-mass work only involves the overall translational and angular displacements, which are familiar to students from their study of kinematics and dynamics.)

(ii) I have implicitly assumed that the spool and table are rigid, so that *rolling* friction can be neglected. This issue may arise after the next point is brought to the attention of students.

(iii) *Static* friction does not change the *thermal* portion of the internal energy of the spool,[17] in striking contrast to *kinetic* friction on a sliding object. This is *not* a consequence of the fact that static friction does zero particle work! For example, the tension also does not change the thermal energy, nor does the static friction on, say, a box that is not slipping in the flat bed of an accelerating truck.

(iv) Nevertheless, static friction does in general alter the *nonthermal* internal energy of the spool, namely its rotational kinetic energy, and this in turn changes the energy available for translation!

We gain additional insights into the physical situation by using the concept of center-of-mass work. Let the system again consist of a single part, the spool. Equation (1) becomes

$$(T-f)L = \tfrac{1}{2}mv^2, \tag{15}$$

while Eq. (4) evaluated about the c.m. of the spool is

$$(Tr+fR)\frac{L}{R}=\tfrac{1}{2}I\omega^2. \tag{16}$$

Adding these two equations together reproduces Eq. (13) and hence the solution Eq. (14). But we now also learn some things about friction. If the frictional force is directed backward (as in Fig. 1), the negative sign on the left-hand side of Eq. (15) implies that friction slows the spool down translationally (compared to what would occur if the coefficient of friction were zero), while the positive sign on the left-hand side of Eq. (16) indicates that it simultaneously speeds the object up rotationally. Furthermore, by multiplying Eq. (15) by γ and equating it to Eq. (16), the frictional force is found to be

$$f=\frac{\gamma-r/R}{\gamma+1}T. \tag{17}$$

(Dividing the absolute value of this result by mg gives the minimum value of the coefficient of static friction if the spool is not to slip for a given pulling force.) Note that the mass distribution factor can be altered within the range $0 < \gamma < 1$. In particular, if the center of the spool is made heavy enough, the frictional force becomes negative, indicating that it is *in the direction of motion*.[18] In that case, friction speeds the object up translationally (at the expense of its rotations) so that this spool would *outpace* an identical one being pulled with the same tension on an air table!

Problem 2: Pushing on a deformable system

Suppose two rigid blocks of mass m are at rest on a level, frictionless surface and are connected by a massless spring of stiffness constant k that is initially relaxed with length L. A constant inward force F is suddenly applied to block 1, displacing it a distance x_1 in the direction of the second block. During this time, the spring coupling causes block 2 to move a distance x_2 in the same direction, as sketched in Fig. 2. (a) Find the resulting velocity of the center of mass, $v_{\text{c.m.}}$, of the pair of blocks. (b) Find the total vibrational energy (kinetic plus potential) of the system, E_{vib}. (c) Find the instantaneous velocity of each block, v_1 and v_2. (d) If the force F is now removed, describe the subsequent motion of the system.

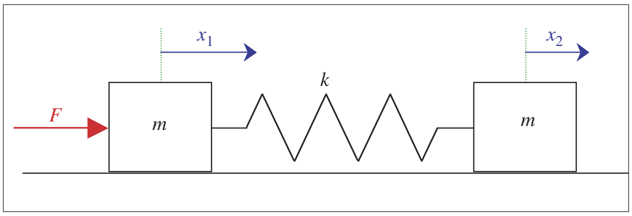

Fig. 2. Free-body diagram of a pair of equal masses *m* connected by an ideal spring *k* and resting on a level, frictionless table. A constant force *F* is applied to the left-hand mass, causing it to translate a distance x_1 rightward while the right-hand mass moves x_2 rightward.

The entire system viewed as one part has mass $2m$. The net force acting on it is F while the c.m. moves a distance of $(x_1 + x_2)/2$, beginning from rest and ending with speed $v_{\text{c.m.}}$. Hence Eq. (1) implies

$$F\,\tfrac{1}{2}(x_1 + x_2) = \tfrac{1}{2}(2m)v_{\text{c.m.}}^2, \tag{18}$$

which immediately gives the solution to (a). On the other hand, Eq. (8) becomes

$$Fx_1 = \tfrac{1}{2}(2m)v_{\text{c.m.}}^2 + E_{\text{vib}}, \tag{19}$$

thereby solving (b). The nonthermal internal energy of the system is $E_{\text{vib}} = 2(\tfrac{1}{2}mv_{\text{rel}}^2) + \tfrac{1}{2}k(x_2 - x_1)^2$ since by symmetry the two blocks have equal and opposite velocities v_{rel} relative to the c.m. This can be solved for v_{rel} and thus the velocities of each block can be calculated from $v_1 = v_{\text{c.m.}} + v_{\text{rel}}$ and $v_2 = v_{\text{c.m.}} - v_{\text{rel}}$, which is (c). As a check, one can apply Eq. (7) or (8) with the system treated mesoscopically as having two parts, namely the two blocks, to obtain

$$Fx_1 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}k(x_2 - x_1)^2. \tag{20}$$

Substitution of the above forms for v_1 and v_2 reproduces Eq. (19). Finally, with respect to (d), note that the c.m. moves with constant velocity $v_{\text{c.m.}}$ after the force F is removed. So jump into the center-of-mass frame. Here one sees that the length of the spring oscillates sinusoidally about L with an angular frequency of $\sqrt{2k/m}$ and an amplitude of $\sqrt{2E_{\text{vib}}/k}$.

This setup is analogous to two massive pistons that can slide frictionlessly along a horizontal pipe with an ideal gas between them. If the far piston is instead clamped in place and the near piston is pushed toward it, this becomes a traditional thermodynamics problem. The compression can be either reversible or irreversible depending on whether the pushing force per unit area of the movable piston is infinitesimally or arbitrarily larger than the gas pressure (and hence than the clamping force per unit area of the fixed piston), respectively. This in turn will determine via Eq. (1) whether the gas will acquire bulk kinetic energy in addition to the change in its internal energy.[19]

Conclusions

Work is always defined as a force integrated over a displacement. However, students must be brought to consciously consider *which* forces and displacements are involved. Depending on the context, they might be asked whether they are including (i) internal or external, (ii) conservative or nonconservative, (iii) field or contact, and (iv) random (thermal) or organized forces in their calculations. In the case of the displacements, the relevant options are (i) center of mass versus point of application, and (ii) translational versus angular.

It helps to explicitly point out that for a particle, $W_{\text{c.m.}} = W_{\text{particle}}$. It is only for objects that can rotate, deform, or undergo irreversible changes that center-of-mass and particle work provide distinct and complementary information about the behavior of a system.[20] Broadly speaking, center-of-mass work relates to the bulk kinetic and potential energies of a system. This is primarily of interest in mechanics problems (particularly when direct solution of Newton's laws would prove difficult). For example, center-of-mass work tells us that a net external force (usually static friction) is needed if a car is to accelerate along a level road. On the other hand, particle work is useful when we are seeking to account for the sources and sinks of energy. Returning to the same example, it is the internal energy of the gasoline that powers the car along the road. Together then, $W_{\text{c.m.}}$ and W_{particle} give us a balanced view of the mechanical universe, and both should be presented in an introductory course.

Acknowledgments

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References

1. For example, see A.J. Mallinckrodt and H.S. Leff, "All about work," *Am. J. Phys.* **60**, 356–365 (April 1992).
2. In partial agreement with R.C. Hilborn, "Let's ban work from physics!" *Phys. Teach*. **38**, 447 (Oct. 2000), I encourage a careful verbal distinction between different kinds of work and an avoidance of the unsubscripted symbol *W.*
3. See the discussion of the "point-particle system" in R.W. Chabay and B.A. Sherwood, *Matter & Interactions I: Modern Mechanics* (Wiley, New York, 2002), Chap. 7.
4. In this context, objects are assumed to be classical and to have mass. Thus, gravitational and electromagnetic fields, and massless particles, springs, strings, and rods are excluded; their role is limited to mediating the interactions between objects.
5. A.B. Arons, "Development of energy concepts in introductory physics courses," *Am. J. Phys*. **67**, 1063–1067 (Dec. 1999).
6. This corresponds to a high-temperature specific heat per atom of $3k_B = 4.14 \times 10^{-23}$ J/K (known as the Dulong-Petit rule), evident, for example, in Fig. 3 of R.W. Chabay and B.A. Sherwood, "Bringing atoms

into first-year physics," *Am. J. Phys.* **67**, 1045–1050 (Dec. 1999).

7. B.A. Sherwood, "Pseudowork and real work," *Am. J. Phys.* **51,** 597–602 (July 1983).
8. A particle is any object having no accessible internal degrees of freedom. It need not be microscopic. The rigid, isothermal, smooth, nonrotating blocks of introductory physics are a macroscopic example.
9. There are no dissipative forces at the particle level. If there were, it would be possible for internal forces to change the energy of an isolated system! (Internal forces can change the net *mechanical* energy of two interacting macroscopic parts, but only with a compensating change in the *internal* energy of these same parts.) For example, the prototypical nonconservative force, friction, actually results from a sum over (conservative) electrostatic forces between the electron shells of the atoms in the contacting surfaces. In consequence, what is called internal work in W.H. Bernard, "Internal work: A misinterpretation," *Am. J. Phys*. **52,** 253–254 (March 1984) is simply the negative of the change in the total (bulk plus internal) potential energy of the system.
10. Internal energy as used here is not identical with what thermodynamics texts usually refer to as internal energy. In the present context, it includes the energy of bulk rotations and other nonthermalized internal modes. See M. Alonso and E.J. Finn, "On the notion of internal energy," *Phys. Educ*. **32**, 256–264 (July 1997).
11. Stored energy is traditionally divided into categories called "forms of energy" such as electric field energy, gravitational potential energy, chemical energy, and so on. But as argued in G. Falk, F. Herrmann, and G.B. Schmid, "Energy forms or energy carriers?" *Am. J. Phys.* **51,** 1074–1077 (Dec. 1983), why should one distinguish forms of energy any more than, say, forms of charge? It is only the *carriers* of the energy or charge, and not the energy or charge *per se*, that change.
12. For example, H. Erlichson, "Internal energy in the first law of thermodynamics," *Am. J. Phys.* **52**, 623–625 (July 1984) distinguishes work from heat on the basis of whether the force is macroscopic or microscopic. With the growing prevalence of microelectromechanical systems (MEMS), I think most physicists would be uncomfortable with this. A somewhat more common approach is that of B.A. Waite, "A gas kinetic explanation of simple thermodynamic processes," *J. Chem. Educ.* **62,** 224–227 (March 1985), who distinguishes work and heat on the basis of whether the interaction is organized or random. While the logic behind this is evident for gases being acted upon by pistons or bunsen burners, it is far less helpful when thinking about dissipation via electrical resistance, kinetic friction, turbulent stirring, and the like.
13. M.W. Zemansky, "The use and misuse of the word 'heat' in physics teaching," *Phys. Teach*. **8**, 295–300 (Sept. 1970). Also see R.H. Romer, "Heat is not a noun," *Am. J. Phys.* **69**, 107–109 (Feb. 2001); R.P. Bauman, "Physics that textbook writers usually get wrong: II. Heat and energy," *Phys. Teach*. **30,** 353–356 (Sept. 1992); G.M. Barrow, "Thermodynamics should be built on energy—Not on heat and work," *J. Chem. Educ.* **65**, 122–125 (Feb. 1988); and M. Tribus, "Generalizing the meaning of 'heat'," *Int. J. Heat Mass Transfer* **11,** 9–14 (Jan. 1968).
14. H. Erlichson, "Are microscopic pictures part of macroscopic thermodynamics?" *Am. J. Phys*. **54**, 665 (July 1986) and S.G. Canagaratna, "Critique of the treatment of work," *Am. J. Phys.* **46,** 1241–1244 (Dec. 1978).
15. B.A. Sherwood and W.H. Bernard, "Work and heat transfer in the presence of sliding friction," *Am. J. Phys.* **52,** 1001–1007 (Nov. 1984). Also see the broken tailhook spring model in R.P. Bauman, "Physics that textbook writers usually get wrong: I. Work," *Phys. Teach.* **30,** 264–269 (May 1992).
16. While the c.m. of the spool translates by L, the spool rotates through an angle of $\theta = L/R$. During this time, a length $l = r\theta$ of string is unwound. The total displacement of point C is thus $L + l = L(1 + r/R)$. Note that this is consistent with the familiar results that point A (equivalent to $r = -R$) has zero net velocity, point B ($r = 0$) has the c.m. velocity v, and point D ($r = R$) has twice the c.m. velocity.
17. As noted in C. Carnero, J. Aguiar, and J. Hierrezuelo, "The work of the frictional force in rolling motion," *Phys. Educ.* **28**, 225–227 (July 1993), *nonconservative* forces need not be *dissipative*.
18. C.E. Mungan, "Acceleration of a pulled spool," *Phys. Teach.* **39,** 481–485 (Nov. 2001).
19. C.E. Mungan, "Irreversible adiabatic compression of an ideal gas," *Phys. Teach.* **41,** 450–453 (Nov. 2003).
20. H.S. Leff and A.J. Mallinckrodt, "Stopping objects with zero external work: Mechanics meets thermodynamics," *Am. J. Phys.* **61,** 121–127 (Feb. 1993).

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