edge-up flotation behavior of the prisms is due to the fact, that they are deprived of one of their degrees of freedom: Their fourfold axis is constrained to stay parallel to the liquid surface.

After these examples it should be dalliance for the dauntless reader to determine the equilibrium positions, as functions of r, of the two remaining Platonic solids: the icosahedron and the pentagondodecahedron.

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APPENDIX A

Explicit expressions for the integral I of Eq. (6) in the cases needed in the analytic calculation:

(1) If $a \geqslant 1$, $b \geqslant 1$, and $c \geqslant 1$:

$$I = -\frac{1}{24abc}\frac{1}{\sqrt{a^2+b^2+c^2}}. \tag{A1}$$

(2) If $a \leqslant 1$, $b \leqslant 1$, $c \leqslant 1$, $a+b \geqslant 1$, $b+c \geqslant 1$, $a+c \geqslant 1$:

$$I = \frac{1}{\sqrt{a^2+b^2+c^2}}\frac{1}{24abc}[(a-1)^4 + (b-1)^4 + (c-1)^4 - 1]. \tag{A2}$$

(3) If $a \leqslant 1$, $b \leqslant 1$, $c \geqslant 1$, $a+b \geqslant 1$:

$$I = \frac{1}{\sqrt{a^2+b^2+c^2}}\frac{1}{24abc}[a^4+b^4-4(a^3+b^3) + 6(a^2+b^2) - 4(a+b) + 1]. \tag{A3}$$

In this case, the immersed volume is

$$V_c = (1/6ab)[1-(1-a)^3-(1-b)^3].$$

There are five other cases, for which we do not give I. They were used in the preliminary numerical work (see Sec. II).

[1]Part 1 of this paper. Equations of Part 1 are referenced by the prefix 1;.

[2]Ch. Dupin, "De la stabilité des corps flottants," 1814, included in *Applications de Géométrie et de Mécanique* (Bachelier, Paris, 1822).

[3]A. S. Ramsey, *Hydrostatics* (Cambridge U.P., Cambridge, 1936), p. 82.

[4]H. Lamb, *Statics* (Cambridge U.P., Cambridge, 1946), 5th ed., p. 421.

[5]H. Yeh and J. I. Abrams, *Principles of Mechanics of Solids and Fluids* (McGraw-Hill, New York, 1960), Vol. 1, p. 96.

[6]E. N. Gilbert, "How things float," Am. Math. Monthly **98**, 201–216 (1991).

All about work

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A comprehensive "taxonomy of work" is developed to clarify the confusing *potpourri* of worklike quantities that exists in the literature. Seven types of work that can be done on a system of particles interacting internally and/or with its environment are identified and reviewed. Each work is defined in terms of relevant forces and displacements; mathematical connections between the works are delineated; work-energy relationships are derived; and the Galilean transformation properties of the works and corresponding energy changes are obtained. The results are applied to several examples, illustrating subtle distinctions between the various works and showing how they can be used to bridge the conceptual gap between the "pure" mechanics of point particles and the thermodynamics of macroscopic matter.

I. INTRODUCTION

It is widely appreciated that the definition of work encountered in most introductory physics textbooks,

$$W \equiv \int \mathbf{F}\cdot d\mathbf{r}, \tag{1}$$

is well defined only when the force acts either on a point particle or a rigid body in pure translation.[1] In the realm of "pure" mechanics—that part of mechanics dealing with nondissipative entities involving observable point particles—the use of Eq. (1) is straightforward.

In contrast, when mechanics is extended into a "real-life" domain involving macroscopic objects with hidden internal energy modes and dissipation, Eq. (1) is inadequate. Authors have invented a potpourri of worklike quantities (hereafter referred to simply as "works") to deal with these situations.[1-13] These works go by a variety of

names including internal work, external macroscopic work, external microscopic work, pseudowork (also called center-of-mass work), conservative work, nonconservative work, quasistatic work, thermodynamic work, and others. The existence of this veritable zoo of works raises several questions: How many types of works are needed? What are the relationships between the works? What work-energy relationships exist? Which works are frame dependent and which are invariant under Galilean transformations? Can the classical mechanics of a many-particle system lead to an understanding of dissipative processes for macroscopic objects?

Inspired by these questions and confounded by the apparent confusion about the definitions for and relationships between the various works, we present a systematic examination of processes in which a classical, many-particle system undergoes mechanical interactions both between its own component particles and with its environment. We provide explicit, microscopically based definitions of seven works that can, in principle, be calculated for such processes. From those definitions we derive the relationships between the works and a set of "work-energy relationships" that connect each work to a change in some form of system energy. Finally, we determine the Galilean transformation characteristics of all works and energy changes.

Although this article is, in part, a review, its principal value is as a needed synthesis and generalization of previous efforts. Using a consistent notation and terminology, defining all relevant quantities unambiguously, and obtaining the relationships between them, we provide, in essence, the fundamentals of a complete "taxonomy of work."

Our investigation is organized as follows: We begin in Sec. II with a survey of prior applications of work concepts to macroscopic objects. Sections III through VI comprise the development of our "standard taxonomy of work." In Sec. VII we focus attention on a collection of engaging applications chosen to illuminate subtle distinctions between the works and linkages between "pure" mechanics of point particles and thermodynamics of macroscopic matter. We close with a summary of our findings and a discussion of the place of this study within the larger context of thermodynamics. Readers who wish to scan the main results, bypassing the details, are directed to Table I and Sec. VIII.

II. SURVEY OF THE LITERATURE

Much of the confusion surrounding the subject of work arises from the lack of a standard and distinct notation system for quantities that are similar enough to convey the unfortunate and mistaken impression that they are identical. The collection of authoritative articles referenced below illustrates this point. Individually, each employs notation that is internally consistent and well suited to the specific investigation being conducted. Between articles, however, one may find the same symbol representing fundamentally different quantities and/or different symbols representing identical quantities. Our notation-free discussion in this section is intended to illustrate the substance and scope of previous efforts while avoiding notational confusion.

Difficulties encountered by naively applying the work-energy theorem to nonrigid and/or rotating bodies have been the subject of articles by various authors.[2-13] Erlichson,[2] Penchina,[3] and Sherwood[4,5] have pointed out that many interactions are best described in terms of a "pseudowork-kinetic energy" theorem. Pseudowork is calculated as the work that *would* be done by a force equal to the net force acting on the system *if* it acted along the path followed by the system's center-of-mass (c.m.) and can be shown to equal the change in bulk translational kinetic energy of the system. It is not real work in a fundamental sense because it generally entails forces that actually act at points distinct from the c.m. through displacements that can differ from that of the c.m.

In a recent review, "Developing the Energy Concepts in Introductory Physics," Arons[6] argues forcefully that "The principal misconception planted in introductory physics is that the 'work' quantity appearing in the 'work-kinetic energy theorem' ... obtained by integration of Newton's Second Law, is identical with the 'work' appearing in the general law of conservation of energy, namely the First Law of Thermodynamics." Following Penchina[3] and Sherwood,[4,5] Arons recommends the use of the term "pseudowork" for the quantity connected to displacement of the c.m., "reserving the name 'work' for the quantity appearing in the First Law of Thermodynamics." He states further, "...it is convenient because it does not completely sever the connection between the two quantities and because it does not resort to a radically new vocabulary."

As pointed out by Arons, careless use of the work-energy equation can lead to mistakes and misconceptions. To illustrate the kinds of difficulties that can occur, consider a completely inelastic collision between two identical masses. Naive application of the standard work-kinetic energy theorem gives the result that the work done in stopping each object is $-\frac{1}{2}mv^2$, where m is the mass and v the initial speed of each object. Unfortunately, this is misleading because a straightforward symmetry argument[7] shows that the work done by each object on the other must be identically zero! A proper treatment of this problem requires one to take into account the energy of deformation.

Bernard[8] and Erlichson[9] have pointed out that the work-kinetic energy theorem can be applied to interactions in general if we take the "work" to be that done on a system by both external *and* internal forces and the "kinetic energy" to be that possessed by the system due to both its bulk translational motion *and* the motions of its constituent parts relative to the c.m. (For a solid body this second constituent of the kinetic energy would include energy associated with bulk rotation and with the vibrational kinetic energies of the molecules that make up the body.)

Canagaratna[10] arrived at the same conclusion but put the result in a different form by defining an internal potential energy change that is directly related to the work done by internal forces during a change in the "relative configuration of the constituent parts of the body." An interesting aspect of Canagaratna's work is the careful distinction made between quasistatic and nonstatic work—a concept that is particularly important in macroscopic mechanics which, in essence, is thermodynamics.

Kemp[11] has argued that difficulties related to the bookkeeping of internal work can be circumvented entirely by appeal to the first law of thermodynamics in the form Δ(mechanical energy of system and surroundings) $+ \Delta$(internal energy of system) $+ \Delta$(internal energy of surroundings) $= 0$. This approach avoids explicit use of work and heat terms, but also conceals the mechanisms responsible for the energy transformations. In a similar

vein Barrow[12] has asserted that "There is no thermodynamic role for the slippery terms 'heat' and 'work.' We should deal with energies and, in thermodynamics, with the energies of the system and its thermal and mechanical surroundings. Then all first-law energy calculations can be done with a good accounting system... ."

Sherwood and Bernard[13] proposed use of two forms of the first law of thermodynamics—one frame dependent and one frame invariant. In the frame-dependent version the work is calculated using all external forces and the motions of their points of application and the system energy includes bulk translational kinetic energy. In the frame-invariant version the work is calculated using all external forces and the motions of their points of application *relative to the system c.m.* and the system energy does *not* include bulk translational kinetic energy. (Note that this second work is calculated in a frame that is not, in general, inertial.) The difference between the two works turns out to be precisely the pseudowork, which equals the change in bulk translational kinetic energy.

The abundance of papers devoted to energy transformations and, in particular, work in macroscopic systems is impressive and is indicative of the discomfort many physics teachers experience in this area. It provides evidence that further clarifications are needed. The "taxonomy of work" presented in Secs. III–VI is intended to meet this need.

III. ASSUMPTIONS AND ELEMENTARY DEFINITIONS

We consider a system of particles as shown in Fig. 1, which may comprise either a rigid or deformable body, and make the following assumptions.

Assumption 1: Our system consists of a collection of N elements, each of which behaves as a point particle. The significance of this assumption is twofold: First, since acceleration is strictly well defined only for point particles, only point particles can strictly obey Newton's second law. Second, point particles, by definition, are devoid of internal structure and are incapable of possessing internal modes that store energy.

Assumption 2: The interelement forces are conservative. This enables us to define an interelement, internal potential energy function for the system.

Assumption 3: A classical, nonrelativistic analysis is adequate. This assumption justifies our use of the simple Galilean transformation and allows us to declare that all forces and time intervals are invariant with respect to changes of reference frame.

For the sake of notational clarity and precision, we collect here the definitions of elementary quantities that are used throughout the paper.

The element i has mass m_i. The position (i.e., displacement from the origin) and velocity of element i in a chosen laboratory frame are denoted, respectively, by $\mathbf{r}_i$ and $\mathbf{v}_i \equiv d\mathbf{r}_i/dt$. Each element i experiences a *net external* force $\mathbf{F}_i^{\text{ext}}$ due to its interactions with agents outside of the system. In addition it experiences *internal* forces $\mathbf{F}_{ij}$ due to its interactions with the other elements $j \neq i$ in the system. The *net internal* force on the ith element is

$$\mathbf{F}_i^{\text{int}} \equiv \sum_{j \neq i} \mathbf{F}_{ij}.$$

The *net* force on element i is $\mathbf{F}_i^{\text{tot}} \equiv \mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$.

The total mass of the N-element system is $M \equiv \Sigma_i m_i$. The position and velocity of its c.m. are denoted, respectively, by

$$\mathbf{r}_{\text{c.m.}} \equiv \frac{1}{M}\sum_i m_i \mathbf{r}_i \quad \text{and} \quad \mathbf{v}_{\text{c.m.}} \equiv \frac{d\mathbf{r}_{\text{c.m.}}}{dt} = \frac{1}{M}\sum_i m_i \mathbf{v}_i.$$

The net force on the system is the sum of the net external and net internal forces; i.e.,

$$\mathbf{F}_{\text{tot}} \equiv \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}} \equiv \sum_i \mathbf{F}_i^{\text{ext}} + \sum_i \mathbf{F}_i^{\text{int}}.$$

However, because

$$\mathbf{F}_{\text{int}} = \sum_i \mathbf{F}_i^{\text{int}} = \sum_i \sum_{j \neq i} \mathbf{F}_{ij} = \sum_i \sum_{j > i} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) = 0$$

by Newton's third law, we have $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{ext}}$.

The position and velocity of element i *relative to the system c.m.* are denoted, respectively, by $\tilde{\mathbf{r}}_i \equiv \mathbf{r}_i - \mathbf{r}_{\text{c.m.}}$ and $\tilde{\mathbf{v}}_i \equiv d\tilde{\mathbf{r}}_i/dt = \mathbf{v}_i - \mathbf{v}_{\text{c.m.}}$. Finally, the position of the ith element *relative to the jth element* is denoted by $\tilde{\mathbf{r}}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j = \tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j$.

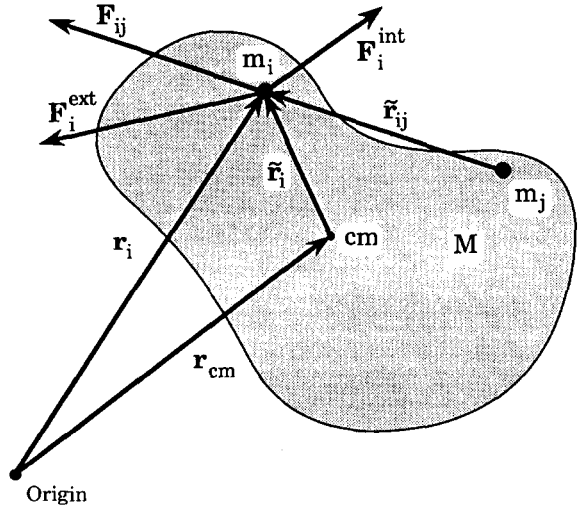

Fig. 1. Depiction of the N-element system on which work is done. Element i has mass m_i, the system has total mass M, and the c.m. of the system has position vector $\mathbf{r}_{\text{c.m.}}$ relative to the origin. The position vectors of element i relative to the origin, to the c.m., and to element j, respectively, are $\mathbf{r}_i$, $\tilde{\mathbf{r}}_i$, and $\tilde{\mathbf{r}}_{ij}$. The force on element i by element j is $\mathbf{F}_{ij}$ and the sum of all such internal forces on element i is $\mathbf{F}_i^{\text{int}}$. The net external force on element i is $\mathbf{F}_i^{\text{ext}}$. Although this figure depicts a continuous collection of elements resembling a rigid body, the mathematical development assumes neither continuity nor rigidity.

IV. WORK DEFINITIONS

We now consider a process in which the system elements interact with each other and/or the external environment and change their positions relative to each other and/or the laboratory frame. The laboratory frame is inertial but otherwise arbitrarily chosen. The process is understood to take place during a well-defined "interaction time period" τ over which all integrals are carried out.

We begin with a definition of the *total* work done on the system—the sum of the works done on each of its elements:

$$W_{\text{tot}} \equiv \sum_i \int \mathbf{F}_i^{\text{tot}} \cdot d\mathbf{r}_i. \tag{2}$$

Note that the total work will, in general, include nonzero contributions from internal forces. We may consider the work done by external and internal forces separately and define, respectively, the *external* and *internal* works:

$$W_{\text{ext}} \equiv \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i, \tag{3}$$

$$W_{\text{int}} \equiv \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d\mathbf{r}_i. \tag{4}$$

Equations (2)–(4) define the three works, W_{tot}, W_{ext}, and W_{int}, which we characterize as "frame specific" (as opposed to "frame dependent"—a term we reserve for characterizing the transformation properties of various quantities) because each involves integrals over the paths of the system elements in the inertial laboratory reference frame. (We shall see that although W_{int} is frame specific, it is *not* frame dependent.)

A fourth worklike quantity is the pseudowork:

$$W_{\text{ps}} \equiv \sum_i \int \mathbf{F}_i^{\text{tot}} \cdot d\mathbf{r}_{\text{c.m.}} = \int \mathbf{F}_{\text{tot}} \cdot d\mathbf{r}_{\text{c.m.}} = \int \mathbf{F}_{\text{ext}} \cdot d\mathbf{r}_{\text{c.m.}}. \tag{5}$$

W_{ps} is also "frame specific" because the motion of the c.m. is relative to our chosen laboratory inertial reference frame. (Some authors prefer to call W_{ps} "center-of-mass work," a name we avoid because of its potential for confusion with the works defined in the next paragraph.)

We now define three quantities that are analogous to W_{tot}, W_{ext}, and W_{int}, but which involve displacements relative to a nonrotating frame traveling with the system c.m. Note that this frame is, in general, *not* inertial. These "system-specific" works are defined as follows:

$$w_{\text{tot}} \equiv \sum_i \int \mathbf{F}_i^{\text{tot}} \cdot d\tilde{\mathbf{r}}_i, \tag{6}$$

$$w_{\text{ext}} \equiv \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\tilde{\mathbf{r}}_i, \tag{7}$$

$$w_{\text{int}} \equiv \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d\tilde{\mathbf{r}}_i. \tag{8}$$

The seven works defined in Eqs. (2)–(8) are not independent. The following four relationships are readily derived:

$$W_{\text{tot}} = W_{\text{ext}} + W_{\text{int}}, \tag{9}$$

$$\begin{aligned} W_{\text{ext}} &= \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d(\mathbf{r}_{\text{c.m.}} + \tilde{\mathbf{r}}_i) \\ &= \int \mathbf{F}_{\text{ext}} \cdot d\mathbf{r}_{\text{c.m.}} + \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\tilde{\mathbf{r}}_i \\ &= W_{\text{ps}} + w_{\text{ext}}, \end{aligned} \tag{10}$$

$$\begin{aligned} W_{\text{int}} &= \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d(\mathbf{r}_{\text{c.m.}} + \tilde{\mathbf{r}}_i) \\ &= \int \mathbf{F}_{\text{int}} \cdot d\mathbf{r}_{\text{c.m.}} + \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d\tilde{\mathbf{r}}_i \\ &= w_{\text{int}}, \end{aligned} \tag{11}$$

$$w_{\text{tot}} = w_{\text{ext}} + w_{\text{int}}. \tag{12}$$

Using these four relationships all seven works can be obtained, for instance, from a knowledge of W_{tot}, W_{ps}, and W_{int}. In the next section we show that these three works—and, by extension, all the others—are directly related to changes in distinctly different forms of system energy.

V. WORK-ENERGY RELATIONSHIPS

By definition, work is a function of a mechanical *process*. However, we show here that each of the "works" defined above can be related via a "work-energy relationship" to a change in some purely state-dependent function that we refer to as an "energy." We first derive such relationships for W_{tot}, W_{ps}, and W_{int}.

Starting with Eq. (2) we find,

$$\begin{aligned} W_{\text{tot}} &\equiv \sum_i \int \mathbf{F}_i^{\text{tot}} \cdot d\mathbf{r}_i \\ &= \sum_i \int m_i \frac{d\mathbf{v}_i}{dt} \cdot \mathbf{v}_i \, dt = \sum_i \frac{1}{2} m_i \int d(v_i^2) \\ &= \Delta\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \\ &= \Delta K_{\text{tot}}. \end{aligned} \tag{13}$$

Here we use our assumption that system elements behave as point particles that obey Newton's second law and introduce the definition,

$$\Delta K_{\text{tot}} \equiv \Delta\left(\sum_i \frac{1}{2} m_i v_i^2 \right), \tag{14}$$

for the change in the total kinetic energy of the system during the interaction period. By expressing the velocity of element i in terms of its velocity relative to the c.m.—i.e., by using the relationship $\mathbf{v}_i = \mathbf{v}_{\text{c.m.}} + \tilde{\mathbf{v}}_i$—we find,

$$\begin{aligned} \Delta K_{\text{tot}} &= \Delta\left(\sum_i \frac{1}{2} m_i |\mathbf{v}_{\text{c.m.}} + \tilde{\mathbf{v}}_i|^2 \right) \\ &= \Delta\left(\sum_i \frac{1}{2} m_i [v_{\text{c.m.}}^2 + \tilde{v}_i^2 + 2(\mathbf{v}_{\text{c.m.}} \cdot \tilde{\mathbf{v}}_i)] \right) \\ &= \Delta\left(\sum_i \frac{1}{2} m_i v_{\text{c.m.}}^2 \right) + \Delta\left(\sum_i \frac{1}{2} m_i \tilde{v}_i^2 \right) \\ &\quad + \Delta\left(\sum_i m_i (\mathbf{v}_{\text{c.m.}} \cdot \tilde{\mathbf{v}}_i) \right) \\ &= \Delta\left(\frac{1}{2} M v_{\text{c.m.}}^2 \right) + \Delta\left(\sum_i \frac{1}{2} m_i \tilde{v}_i^2 \right) \\ &\quad + \Delta\left(\mathbf{v}_{\text{c.m.}} \cdot \sum_i m_i \tilde{\mathbf{v}}_i \right) \\ &= \Delta K_{\text{tr}} + \Delta K_{\text{int}}, \end{aligned} \tag{15}$$

where we use the fact that $\Sigma_i m_i \tilde{\mathbf{v}}_i = 0$ (by the definition of c.m.) and introduce the definitions,

$$\Delta K_{\text{tr}} \equiv \Delta\left(\frac{1}{2} M v_{\text{c.m.}}^2 \right), \tag{16}$$

$$\Delta K_{\text{int}} \equiv \Delta\left(\sum_i \frac{1}{2} m_i \tilde{v}_i^2 \right). \tag{17}$$

ΔK_{tr} is the change in the bulk translational kinetic energy of the system—the energy that the system possesses by virtue of the motion of its c.m. relative to the observer. ΔK_{int} is

the change in the internal kinetic energy of the system—the energy that the system possesses as a result of deviations of the velocities of its elements from that of the c.m. These deviations can include the effects of vibration and bulk rotation.

Next, starting with Eq. (5) we find

$$\begin{aligned} W_{ps} &\equiv \sum_i \int \mathbf{F}_i^{tot}\cdot d\mathbf{r}_{c.m.} = \sum_i \int m_i \frac{d\mathbf{v}_i}{dt}\cdot d\mathbf{r}_{c.m.} \\ &= \sum_i \int m_i\left(\frac{d}{dt}\left(\mathbf{v}_{c.m.} + \tilde{\mathbf{v}}_i\right)\right)\cdot d\mathbf{r}_{c.m.} \\ &= \sum_i m_i \int \frac{d\mathbf{v}_{c.m.}}{dt}\cdot \mathbf{v}_{c.m.}\, dt + \sum_i \int m_i \frac{d\tilde{\mathbf{v}}_i}{dt}\cdot d\mathbf{r}_{c.m.} \\ &= \frac{1}{2} M \int d(v_{c.m.}^2) + \int \left(\frac{d}{dt}\sum_i m_i \tilde{\mathbf{v}}_i\right)\cdot d\mathbf{r}_{c.m.} \\ &= \Delta\left(\tfrac{1}{2} M v_{c.m.}^2\right) \\ &= \Delta K_{tr}. \end{aligned} \tag{18}$$

This is the well-known pseudowork-kinetic energy relationship.

Finally, starting with Eq. (4) we find,

$$\begin{aligned} W_{int} &\equiv \sum_i \int \mathbf{F}_i^{int}\cdot d\mathbf{r}_i = \sum_i \sum_{j\neq i} \int \mathbf{F}_{ij}\cdot d\mathbf{r}_i \\ &= \sum_{\text{all pairs}} \left(\int \mathbf{F}_{ij}\cdot d\mathbf{r}_i + \int \mathbf{F}_{ji}\cdot d\mathbf{r}_j\right) \\ &= \sum_{\text{all pairs}} \int \mathbf{F}_{ij}\cdot d(\mathbf{r}_i - \mathbf{r}_j) \\ &= \sum_{\text{all pairs}} \int \mathbf{F}_{ij}\cdot d\tilde{\mathbf{r}}_{ij} \\ &= \sum_{\text{all pairs}} \int dW_{ij} \\ &= \sum_{\text{all pairs}} (-\Delta\Phi_{ij}) = -\Delta\Phi, \end{aligned} \tag{19}$$

where we use Newton's third law ($\mathbf{F}_{ji} = -\mathbf{F}_{ij}$) and our assumption that all internal forces are conservative, and introduce the definitions,

$$dW_{ij} \equiv \mathbf{F}_{ij}\cdot d\tilde{\mathbf{r}}_{ij}, \quad \Delta\Phi_{ij} \equiv -\int dW_{ij}$$

and

$$\Delta\Phi \equiv \sum_{\text{all pairs}} \Delta\Phi_{ij}. \tag{20}$$

$\Delta\Phi$ is the change in the internal potential energy of the system—the energy that the system possesses by virtue of the conservative interaction forces between its elements. In the case of a rigid body these would be the "binding forces" and changes in Φ would result from temporary or permanent deformations. In the usual fashion it is the assumption of conservative internal forces that allows us to assert that each work integral is path independent and to define the potential energy functions Φ_{ij} for each interacting pair of elements.

Equations (13), (18), and (19) are work-energy relationships for three of the seven works defined in the previous section. With the help of the four relationships between the works in Eqs. (9)–(12) we derive the following four additional work-energy relationships:

$$W_{ext} = \Delta E, \tag{21}$$

$$w_{ext} = \Delta U, \tag{22}$$

$$w_{int} = -\Delta\Phi, \tag{23}$$

$$w_{tot} = \Delta K_{int}, \tag{24}$$

in which we introduce the following definitions:

$$\Delta U \equiv \Delta K_{int} + \Delta\Phi, \tag{25}$$

$$\Delta E \equiv \Delta K_{tr} + \Delta U. \tag{26}$$

ΔU is the change in internal energy of the system, which includes both kinetic and potential energy contributions. ΔE is the change in total energy of the system, which includes changes in both the internal energy and the bulk translational kinetic energy. Note carefully that there is no term representing the potential energy of the system with respect to its surroundings since the contributions of *all* external forces—conservative or not—are accounted for on the "work side" of our work-energy relationships.[14]

The seven works and their six associated energy changes are summarized in Table I, which also gives the definition of each quantity, the auxiliary relationships between it and the others, an interpretation of each work-energy relation, and whether it is frame dependent or frame invariant. (Transformation properties are discussed in the next section.)

We reiterate that among the seven works and six energy changes there are only three independent quantities. Although many sets of "basis" quantities might be chosen, perhaps the most appropriate candidates are: (1) ΔK_{tr}, which is simply associated with the choice of laboratory frame; (2) ΔK_{int}, which characterizes the change in the internal state of motion of the system; and (3) $\Delta\Phi$, which characterizes the change in the system's configurational energy. Taking these three quantities as our basis we may summarize the relationships between the quantities in matrix form:

$$\begin{pmatrix} W_{tot} \\ W_{ext} \\ W_{int} \\ W_{ps} \\ w_{tot} \\ w_{ext} \\ w_{int} \\ \Delta K_{tot} \\ \Delta U \\ \Delta E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Delta K_{tr} \\ \Delta K_{int} \\ \Delta\Phi \end{pmatrix}. \tag{27}$$

VI. TRANSFORMATION CHARACTERISTICS

In this section we investigate the transformation characteristics of the previously defined works and energy changes. As indicated by Eq. (27), we need only determine the transformation characteristics of ΔK_{tr}, ΔK_{int}, and $\Delta\Phi$ because all other quantities can be obtained in terms of these. We determine the values of these energies in a "u frame" traveling with velocity $\mathbf{u}$ with respect to the laboratory frame. The interelement forces are independent of reference frame by assumption and the Galilean transformation to the u frame yields,

Table I. Seven works with definitions, equivalent energy changes, interrelationships, interpretations, and frame-dependence properties. Each of the seven works and six energies is precisely defined in terms of the masses, trajectories, and velocities of the particles that make up the system and the forces that act on them during the interaction process.

Work	Associated energy change	Interpretation and frame dependence
$W_{\rm tot} \equiv \sum_i \int \mathbf{F}_i^{\rm tot} \cdot d\mathbf{r}_i$	$\Delta K_{\rm tot} \equiv \Delta\left(\sum_i \frac{1}{2} m_i v_i^2\right)$	The frame-specific total work is equal to the change in the total kinetic energy.
$= W_{\rm ext} + W_{\rm int}$	$= \Delta K_{\rm tr} + \Delta K_{\rm int}$	Frame dependent.
$W_{\rm ext} \equiv \sum_i \int \mathbf{F}_i^{\rm ext} \cdot d\mathbf{r}_i$	$\Delta E \equiv \Delta K_{\rm tr} + \Delta U$	The frame-specific external work is equal to the change in the total energy.
$= W_{\rm ps} + w_{\rm ext}$	$= \Delta K_{\rm tr} + \Delta K_{\rm int} + \Delta\Phi$	Frame dependent.
$W_{\rm int} \equiv \sum_i \int \mathbf{F}_i^{\rm int} \cdot d\mathbf{r}_i$	$-\Delta\Phi \equiv \sum_{\text{all pairs}} \int \mathbf{F}_{ij} \cdot d\tilde{\mathbf{r}}_{ij}$	The frame-specific internal work is equal and opposite to the change in the internal potential energy.
$= w_{\rm int}$		Frame invariant.
$W_{\rm ps} \equiv \sum_i \int \mathbf{F}_i^{\rm tot} \cdot d\mathbf{r}_{\rm c.m.}$	$\Delta K_{\rm tr} \equiv \Delta\left(\frac{1}{2} M v_{\rm c.m.}^2\right)$	The pseudowork is equal to the change in the bulk translational kinetic energy.
		Frame dependent.
$w_{\rm tot} \equiv \sum_i \int \mathbf{F}_i^{\rm tot} \cdot d\tilde{\mathbf{r}}_i$	$\Delta K_{\rm int} \equiv \Delta\left(\sum_i \frac{1}{2} m_i \tilde{v}_i^2\right)$	The system-specific total work is equal to the change in the internal kinetic energy.
$= w_{\rm ext} + w_{\rm int}$		Frame invariant.
$w_{\rm ext} \equiv \sum_i \int \mathbf{F}_i^{\rm ext} \cdot d\tilde{\mathbf{r}}_i$	$\Delta U \equiv \Delta K_{\rm int} + \Delta\Phi$	The system-specific external work is equal to the change in the total internal energy.
		Frame invariant.
$w_{\rm int} \equiv \sum_i \int \mathbf{F}_i^{\rm int} \cdot d\tilde{\mathbf{r}}_i$	$-\Delta\Phi$	The system-specific internal work is equal and opposite to the change in the internal potential energy.
		Frame invariant.

$$\mathbf{v}'_{\rm c.m.} = \mathbf{v}_{\rm c.m.} - \mathbf{u}, \quad \tilde{\mathbf{v}}'_i = \tilde{\mathbf{v}}_i, \quad \text{and} \quad \tilde{\mathbf{r}}'_{ij} = \tilde{\mathbf{r}}_{ij}. \tag{28}$$

Therefore,

$$\Delta K'_{\rm tr} = \Delta\left(\tfrac{1}{2} M |\mathbf{v}_{\rm c.m.} - \mathbf{u}|^2\right)$$
$$= \Delta K_{\rm tr} - \mathbf{u}\cdot\Delta\mathbf{P}, \tag{29}$$

where $\mathbf{P} \equiv M\mathbf{v}_{\rm c.m.}$, the linear momentum of the system. Of course, $\Delta\mathbf{P}$ equals the impulse $\mathbf{I} = \int \mathbf{F}_{\rm ext}\, dt$ on the system during the interaction period. The impulse and, therefore, the momentum change $\Delta\mathbf{P}$ are manifestly invariant with respect to Galilean changes of reference frame. Equation (29) makes it clear that $\Delta K_{\rm tr} \neq \Delta K'_{\rm tr}$ unless $\mathbf{u}\cdot\Delta\mathbf{P} = 0$. Proceeding:

$$\Delta K'_{\rm int} = \Delta\left(\sum_i \frac{1}{2} m_i \tilde{v}_i'^2\right) = \Delta K_{\rm int} \tag{30}$$

and

$$\Delta\Phi' = -\sum_{\text{all pairs}} \int \mathbf{F}_{ij} \cdot d\tilde{\mathbf{r}}'_{ij} = \Delta\Phi. \tag{31}$$

Thus $\Delta K_{\rm int}$ and $\Delta\Phi$ are frame invariant while $\Delta K_{\rm tr}$ is frame dependent.

Equations (29)–(31) demonstrate that, of our three chosen basis quantities, only $\Delta K_{\rm tr}$ is frame dependent. As a result, only those works and energy changes that show a dependence on $\Delta K_{\rm tr}$ in Eq. (27)—i.e., $W_{\rm tot}$, $W_{\rm ext}$, $W_{\rm ps}$, $\Delta K_{\rm tot}$, and ΔE—will themselves be frame dependent. Furthermore, because of their simple additive dependence on $\Delta K_{\rm tr}$, each will transform in a manner identical to Eq. (29).

Two corollaries are immediately evident.

Corollary 1: For any interaction process with $\Delta\mathbf{P} = \mathbf{0}$, *all* works and energy changes are frame invariant and $\Delta K_{\rm tr} = 0$. It then also follows that $W_{\rm ps} = 0$, that $W_{\rm tot} = w_{\rm tot} = \Delta K_{\rm tot} = \Delta K_{\rm int}$, and that $W_{\rm ext} = w_{\rm ext} = \Delta E = \Delta U$.

Corollary 2: For any interaction process in which $\Delta\mathbf{P} \neq \mathbf{0}$, we can find families of reference frames in which any given frame-dependent quantity takes on any desired value including, for instance, zero. These families are de-

termined by the requirement that their frame velocities relative to the lab frame obey $\mathbf{u}\cdot\Delta\mathbf{P}=$ (value of the quantity in the lab frame) − (desired value). This relationship implies that the relative velocity *between* any two frames in such a family is perpendicular to $\Delta\mathbf{P}$.

VII. APPLICATIONS

In this section we consider four examples illustrating the applications of our results.

A. Example 1: Head-on collision

Here we consider a one-dimensional, completely inelastic collision between two objects, each of mass m.[7] In the c.m. frame, their velocities prior to the collision are $+\mathbf{v}$ and $-\mathbf{v}$; after the collision they are both at rest. We take our system to comprise both objects. In *any* frame we find $\Delta K_{\rm tr}=0$ because the c.m. velocity is constant. Furthermore, because there are no external forces acting on the system $W_{\rm ext}=0$ again in any frame. The second row of Table I then shows that $\Delta U=\Delta K_{\rm int}+\Delta\Phi=0$. Notice that $K_{\rm int}$ includes the translational kinetic energies of both objects as well as the system's total microscopic kinetic energy; i.e., $\Delta K_{\rm int}=\Delta K_{\rm tr,1}+\Delta K_{\rm tr,2}+\Delta K_{\rm int,microscopic}$. It is straightforward to show that $\Delta K_{\rm tr,1}+\Delta K_{\rm tr,2}=-mv^2$ in any frame. Therefore, $\Delta K_{\rm int,microscopic}+\Delta\Phi=mv^2$ (in any frame) and we conclude that the effect of the head-on collision is that an amount of energy mv^2 (which was macroscopically *observable* before the collision in the form of the bulk kinetic energy of its two important components) is converted into macroscopically *unobservable* internal potential energy and microscopic kinetic energy. Notice that there is no heat transfer involved in this process—although there might be as a *result* of the process. The energy transformations we are discussing here are calculable, at least in principle, by a purely mechanical theory.

If we define our system to be *one* of the two objects, we find highly frame-dependent values for $W_{\rm ext}$ and $\Delta K_{\rm tr}$ and it becomes more convenient to consider the system-specific works. For instance, since the contact surface with the other object will, predominantly, move toward *and* exert forces toward the system c.m., we find that $w_{\rm ext}=\Delta U>0$; i.e., the internal energy increases due to the compressive forces of the collision. On the other hand, without further details we cannot determine the sign of $w_{\rm int}$ and, therefore, of $\Delta\Phi$. Although we might anticipate increased internal potential energy, the collision may just as well act as the catalyst for its release.

B. Example 2: Free expansion of a gas

Next we consider a gas confined to the left half of an enclosure with the right half entirely empty. The system elements here are gas molecules. The enclosure is assumed to be "perfectly insulating" so that energy transfer between the gas molecules and the environment is negligible. We denote the internal energy of the gas by U_0, view the system in the rest frame of the enclosure, and assume the initial c.m. velocity of the gas is zero. Now suppose the partition separating the left and right halves of the container spontaneously collapses allowing gas molecules to fill the entire enclosure.

With the collapse of the partition, the leftward force previously constraining the molecules to the left half becomes zero. The force from the left wall is still nonzero, so there is a net force directed to the right. Although this force does no real physical work, it does generate positive pseudowork (because the rightward force acts on the gas while its c.m. moves to the right) that is related to the increased bulk translational kinetic energy—i.e., $W_{\rm ps}=\Delta K_{\rm tr}>0$. At all times, however, we have $W_{\rm ext}=0$, so Table I shows that $\Delta U=-\Delta K_{\rm tr}<0$. That is, the bulk kinetic energy is derived from the internal energy of the gas. Furthermore, if the gas is "ideal" in the sense that the intermolecular potential energy $\Phi=0$ (or is at least negligibly small) then $\Delta U=\Delta K_{\rm int}$ and we see that the translational kinetic energy is, in a sense, nothing more than "redirected" internal kinetic energy. The nonzero bulk translational energy of the system is only temporary, however. As the gas expands to fill the enclosure, the new right wall begins to exert a force greater than that from the left wall. This generates negative pseudowork (because a net *left*ward force now acts on the gas while its c.m. is still moving to the right) and decreases the bulk translational kinetic energy. After being brought to rest the c.m. may reaccelerate to the left and the process may repeat as the gas "sloshes" back and forth until the amplitude of the c.m. motion decreases to the level of statistical fluctuations.

It is instructive to look at the process from the standpoint of the system–specific works also. Just after the partition collapses, the left wall moves to the left (from the standpoint of a frame moving with the system c.m.) while exerting a force toward the right and, therefore, performs negative system-specific external work. Table I shows that $w_{\rm ext}=\Delta U=\Delta K_{\rm int}<0$; i.e., the internal energy of the system decreases in agreement with the analysis above. Later, when the right wall begins to exert its large leftward force while moving to the left (again as seen from the system c.m.), positive system-specific external work is done and the internal energy of the system increases again.

Once equilibrium is reached and the net force is again zero, the gas has its initial internal energy $U=U_0$. If the gas is "ideal" this means that $K_{\rm int}=K_{\rm int,0}$. However, the final configuration leaves the N molecules farther apart on average and, if the gas molecules interact according to a typical intermolecular potential energy, this configuration is expected to have an *increased* potential energy Φ. The constancy of U then means that the internal kinetic energy has decreased. We learn in thermodynamics that the "free expansion" of an ideal gas leaves the temperature unchanged while that of an interacting gas leads to a temperature decrease. Thus we find a correlation between temperature and kinetic energy for a classical gas. Although this connection is not necessarily valid for other systems,[15] it is commonly encountered in both kinetic theory and statistical mechanics. The fact that it is suggested by a classical mechanics argument is noteworthy.

C. Example 3: Slow expansions of an ideal gas

Here we consider slow expansions of an ideal gas ($\Phi=0$) under two different special circumstances. In both cases the N-element gas is assumed to be in a horizontal cylinder, the right vertical wall of which is a movable, frictionless piston. The expansions, whereby the piston moves through a distance d, are assumed to be sufficiently slow that the piston's force $\mathbf{F}_{\rm piston}$ (leftward) on the gas is equal to the opposing external force $\mathbf{F}_{\rm fixed}$ (rightward) at

the fixed, left container wall. This infinite slowness means that the interaction time τ becomes arbitrarily large; such processes are called *quasistatic*. Of course, $\mathbf{F}_{\text{piston}}$ and $\mathbf{F}_{\text{fixed}}$ are both expected to decrease in magnitude as the gas expands but this does not affect our analysis.

For the first expansion type we assume that the only energy interactions with the gas are via work done by the piston. Because the molecules of an ideal gas do zero work on each other,

$$W_{\text{tot}} = W_{\text{ext}} = \int_0^d \mathbf{F}_{\text{piston}} \cdot d\mathbf{r} < 0.$$

The inequality holds because the piston force is directed left while its displacement is directed right. Referring to rows 1 and 2 of Table I (which are identical for an ideal gas), this implies $W_{\text{ext}} = \Delta K_{\text{tr}} + \Delta K_{\text{int}} < 0$. Row 4 tells us that $W_{\text{ps}} = 0$ because the net force on the system is identically zero at all times. This implies $\Delta K_{\text{tr}} = 0$, as expected because of the slowness of the piston action. Using the latter result in row 1, we have $W_{\text{ext}} = \Delta K_{\text{int}} < 0$; i.e., the work on the gas is negative and equals the change in its internal energy. In other words, the gas does positive work $(-W_{\text{ext}})$ on the piston at the expense of its internal energy. The piston's energy does not change because it moves arbitrarily slowly. Of course some other agent, which is unspecified here, must receive the energy given up by the gas. In thermodynamics an expansion where the only energy transfer is mechanical is referred to as an *adiabatic* expansion.

For the second expansion type we assume that (somehow) the internal energy of the gas does not change; i.e., $\Delta K_{\text{int}} = 0$. The zero-net-force constraint assures, as in the latter case, that $\Delta K_{\text{tr}} = 0$. Thus row 1 of Table I implies

$$W_{\text{tot}} = W_{\text{ext}} = \int_0^d \mathbf{F}_{\text{piston}} \cdot d\mathbf{r} = \Delta K_{\text{int}} = 0.$$

This contradicts the obvious fact that the integral must have a nonzero (in fact, negative) value.[16] We conclude that *if* it is possible to maintain the constancy of K_{int} during a slow expansion this must come about via nonmechanical means. Referring to the suggestion in Example 2 that the constancy of K_{int} can be related to temperature constancy we are tempted to envision the possibility of a slow, nonadiabatic, isothermal expansion during which a nonmechanical energy transfer occurs. Such expansions are commonplace in thermodynamics where the nonmechanical energy transfer is called *heat*. Again, classical mechanics has led to an important thermodynamic concept.

D. Example 4: Conveyor-belt variant

The standard conveyor-belt problem envisions the continuous flow of mass onto a constant-velocity horizontal belt. Despite its simple description, this problem contains subtleties that have engendered a rather interesting literature.[17] We consider a variant here in which a crate of mass m is placed, with zero velocity in the lab frame, onto a horizontal belt moving with velocity v. As the crate is accelerated to velocity v in time τ, a force F_e is applied to the belt in order to maintain constant speed. Row 2 of Table I shows that, during the acceleration, the external work done on the crate is $W_{\text{ext},c} = \frac{1}{2}mv^2 + \Delta U_c$. Similarly, the external work done on the belt

$$W_{\text{ext},b} = \int F_e \, dx + (-W_{\text{ext},c}) = \Delta K_{\text{tr},b} + \Delta U_b = \Delta U_b$$

because the constant-speed constraint insures that $\Delta K_{\text{tr},b} = 0$.

We assume that the friction force on the crate (and thus on the belt) is constant throughout $(0,\tau)$, whence the above integral reduces to $F_e d = F_e v\tau$ and we find from above that $F_e v\tau - \frac{1}{2}mv^2 = \Delta U_c + \Delta U_b$. The force F_e generates impulse $F_e\tau$ that must equal the change in momentum mv of the crate. Thus $F_e v\tau = mv^2$ and we have the result $\Delta U_c + \Delta U_b = \frac{1}{2}mv^2$. The interpretation here is that the crate cannot be accelerated without the existence of friction and such friction cannot act without being accompanied by energy dissipation. We might account for the energy transformations by saying that half of the work done by F_e is dissipated into internal energy and the other half goes into the bulk translational kinetic energy of the crate. We will see, however, that this view of the overall energy accounting is highly frame dependent.

Consider an analysis of the same situation from the viewpoint of the (inertial) frame of the belt. In this frame $W'_{\text{ext},c} = -\frac{1}{2}mv^2 + \Delta U_c$. The force F_e does zero work in this frame because we assume that the displacement of its point of application is zero. Thus

$$W'_{\text{ext},b} = 0 + (-W'_{\text{ext},c}) = \Delta K'_{\text{tr},b} + \Delta U_b = \Delta U_b.$$

Combining the latter two equations gives $\Delta U_c + \Delta U_b = \frac{1}{2}mv^2$, in agreement with our result in the lab frame. It is also easy to verify that the works on the crate and on the belt transform properly; i.e., $W'_{\text{ext},c} = W_{\text{ext},c} - \mathbf{u}\cdot\Delta\mathbf{P}_c = W_{\text{ext},c} - mv^2$ and $W'_{\text{ext},b} = W_{\text{ext},b} - \mathbf{u}\cdot\Delta\mathbf{P}_b = W_{\text{ext},b}$. However, in contrast with the overall energy accounting given in the lab frame, the source of the dissipated energy (which is frame invariant) now *appears* to be the initial kinetic energy of the crate. These results serve to emphasize the slipperiness of trying to perform this type of energy accounting.

Because the belt's momentum does not change (i.e., $\Delta\mathbf{P}_b = 0$), Corollary 1 (in the previous section) shows that all works on and energy changes of the belt are frame invariant. On the other hand, since $\Delta\mathbf{P}_c \neq 0$, we may apply Corollary 2 to the crate. Consider a frame moving right with velocity $v/2$ relative to the lab frame. In this frame $\Delta K'_{\text{tr},c} = 0$ because the crate begins with velocity $-v/2$ just after it is placed on the belt and ends with velocity $+v/2$ at full speed. The total work done on the crate in this frame is $W'_{\text{tot},c} = W'_{\text{ext},c} + W'_{\text{int},c} = \Delta K_{\text{int},c}$, which is equivalent to saying that $W'_{\text{ext},c} = \Delta U_c$.

We may locate yet another frame with velocity $v/2 + \Delta K_{\text{int},c}/mv$ relative to the lab frame in which the *total* work on the crate vanishes; i.e., $W'_{\text{tot},c} = 0$. It is straightforward to verify that, in this frame, $\Delta K'_{\text{tr},c} = -\Delta K_{\text{int},c}$. This example makes it clear that the total work can be made to vanish simply by choosing our inertial frame of observation such that the bulk translational kinetic energy change is equal and opposite to the (frame invariant) internal kinetic energy change. A frame in which the *external* work vanishes can be found in a similar manner.

VIII. SUMMARY AND CONCLUSIONS

Table I summarizes the main findings of this paper. We return briefly to the questions raised in Sec. I. First, how

many types of works are needed? The answer is that three works are sufficient to generate the set of seven works that have been identified. Second, which works are independent of the others? Because of the four relationships that exist between the seven works (indicated after the definitions of W_{tot}, W_{ext}, W_{int}, and w_{tot} in the first column of Table I), there are only three independent works. Third, what work-energy relationships exist? These are shown by the associated energy changes in the second column of Table I, which also gives the elementary definitions of the energy changes and their interrelationships. Fourth, which works are frame dependent and which are invariant under Galilean transformations? These are indicated by a comment in the third column of Table I and comprise all works and energy changes that show a dependence on the value of ΔK_{tr} in Eq. (27).

Finally, can the classical mechanics of a many-particle system lead to an understanding of dissipative processes for macroscopic objects? The answer here is a conditional "yes." Classical mechanics enables us to associate dissipation with systems that have hidden energy modes—namely the modes associated with the large numbers of molecules that comprise a macroscopic system. It illustrates that macroscopically observable "bulk" energy can be converted into internal energy and makes plausible the experiential fact that the reverse process is less easily achieved because we have no practical way to control the numerous energy transformations among the molecules that make up an object. After all, the reversal of a head-on, totally inelastic collision would require arranging energy transfers among $\sim 10^{23}$ molecules such that they spontaneously exhibited bulk translational motion. This would be quite remarkable!

On the other hand, if the head-on collision is only slightly inelastic then the two colliding objects *do* recover a fraction of their bulk motion as they bounce off each other. In this sense elasticity can be associated with memory. Elastic materials "remember" their initial configurations—at least partially—and facilitate the ordered energy transformations needed to produce bulk motion. Just as elastic processes exhibit memory effects, inelastic processes do the reverse—erasing memory. For example, if a ball is dropped from a height of 1 m, bounces a few times and then comes to rest, there is no way to uniquely reconstruct its initial configuration from an analysis of the internal energy gains of the ball and ground. *Loss* of information regarding the initial condition can be thought of in terms of a *gain* of missing information. This has nothing to do with a particular observer; the information is missing for *any* observer whose power of measurement is limited to typical macroscopic resolution levels. The importance of memory effects in thermodynamics is exemplified by Bennett's innovative analysis of the Maxwell's demon puzzle, whereby the demon is prevented from violating the second law because its memory must be erased.[18] Of course, classical mechanics does not *predict* this loss of information. That requires the apparatus of thermodynamics and/or statistical physics to develop the second law of thermodynamics.

The results of this paper have led us indirectly and incompletely to the concept of heat. Roughly, conductive heat transfer can be viewed as external work performed at the system interface via "collisions" with the elements of neighboring systems. Though appealing, this explanation fails to make a clear distinction between heat transfer and mechanical work at a surface. That distinction can indeed be a difficult one.[19] In thermodynamics, it is sometimes based upon observable macroscopic work processes and sometimes upon heat transfers between objects with measurable temperature differences. While the spirit of classical mechanics is to describe *all* the elements of a system, the spirit of thermodynamics is to deal only with macroscopically observable quantities. Attempts to bridge the gap between these two disciplines easily lead to the simultaneous consideration of both macroscopic and microscopic phenomena. This is conceptually pleasing but, as in the classic case of Maxwell's demon, we must be mindful of what quantities are measurable macroscopically and what quantities are useful only from a conceptual viewpoint.[20]

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Using small, rare-earth magnets to study the susceptibility of feebly magnetic metals

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Small rare-earth magnets have remarkable properties that can be exploited in the construction of simple measuring devices. Examples are a torsion pendulum that measures the earth's horizontal magnetic field and a gravimetric pendulum that responds to the small susceptibilities of various "nonmagnetic" materials. These preliminary demonstrations lead to the design of a practical device for measuring the susceptibilities of feebly magnetic metals in low magnetic fields. Our laboratory has used this last method to determine relatively small magnetic susceptibilities, down to 0.0003, to an uncertainty of less than 20%. A convenient feature of our apparatus is that calibration by a susceptibility standard is not required.

I. INTRODUCTION

Small but powerful rare-earth magnets are readily available from commercial sources. Cylindrical magnets with diameter twice their length are so stable that they are shipped already magnetized and no "keeper" is required. The building blocks of the devices to be discussed below are cylindrical magnets of samarium–cobalt[1] having a diameter of 5 mm and a height of 2.5 mm. They have a magnetization **M** of about 900 kA/m, aligned along the geometric axis of symmetry. Although this magnetization (an intensive quantity) is typical of the most powerful permanent magnets of any size, the magnetic dipole moment **m** (an extensive quantity) is conveniently small.

The small magnetic moment and compact shape of these magnets lead to the following desirable properties: the dipole approximation becomes appropriate at about 10 mm from the geometric center of a magnet;[2] at 10 cm from the magnet, the fields are already far too weak to destroy magnetic coding on credit cards and other tape media; there is little chance of physical injury to the experimenter as attracting magnets are placed together. The last two properties are important safety features, as anyone who has worked with powerful magnets will agree.

Section IV describes a simple device, based on a magnetic dipole, for low-field measurements of the weak magnetic susceptibility of materials such as AISI 304 stainless steel. The two preliminary experiments of Sec. III provide insight into the magnetic dipole and its interaction with feebly magnetic materials. The work presented below was motivated by our need to select metals that are sufficiently nonmagnetic for constructing precision mass balances. This application is also discussed in Sec. IV.

A basic acquaintance with magnetostatics is required in what follows although expertise in the rather specialized discipline of permanent magnets will not be necessary. We have chosen Duffin's *Electricity and Magnetism*[3] as our principal reference in part because it uses SI units throughout. The magnetic dipole plays an important role in our theoretical analysis so we begin with a brief review of its mathematical description.

II. THE MAGNETIC DIPOLE

When a permanent magnet is placed in a uniform **B** field, it is subject to a torque **Γ**:

$$\boldsymbol{\Gamma} = \mathbf{m} \times \mathbf{B}. \tag{1}$$

Equation (1), in Duffin's treatment,[4] defines the magnetic dipole moment **m**. The direction of **m** is that of the **B** field when the freely suspended magnet is in equilibrium. The magnitude of **m** is the torque experienced by the magnet in a **B** field of 1 T when **m** is perpendicular to **B**. Although the units of **m** are evidently N m T^{-1}, it is customary to use the equivalent units A m^2. Duffin defines the magnetization **M** as the magnetic dipole moment per unit volume:[5]

$$d\mathbf{m} = \mathbf{M}\, d\tau, \tag{2}$$

where $d\tau$ is a small volume with magnetic moment $d\mathbf{m}$.

With the magnetic dipole moment defined, we restate the familiar relations in spherical coordinates for the **B** field in vacuum at distances large[2] compared to the magnet dimensions:

$$B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3}; \quad B_\theta = \mu_0 \frac{m\sin\theta}{4\pi r^3}; \quad B_\varphi = 0, \tag{3}$$

where r is the distance from the dipole, θ is the angle formed by r and the dipole axis, and μ_0 is the vacuum per-