Work Reworked

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Students often encounter some difficulties with work and potential energy using the approaches given in most introductory physics texts. For instance, one difficulty that occurs is associated with the question

> **"What is the work done in lifting a book (of mass *m*) from the floor onto a table through a vertical distance *h*?"**

Of course, the work done is *mgh*. But students are naturally curious and ask questions like:

> **"To get the book moving in the first place, surely it has to be accelerated and so don't you have to use a force greater than *mg*?"**

I can (and do) tell them that is true, but a force less than *mg* is required to "slow it down" again, so that the average force is *mg*. But for some students that is not sufficient. Almost all texts analyze this problem in terms of gravitational potential energy. But that doesn't help much because often the chapter on potential energy occurs *after* the chapter dealing with work, and I hesitate to ask students to wait! A lengthy but useful class discussion persuaded me that I should try another approach. As instructors, we are all familiar with the connections between work and energy, but some of the concepts are much less clear to students, particularly those in noncalculus-based courses. I hope the suggestions I outline below will provide instructors who feel the same discomfort I did (!) with some ideas.

So, here are two alternative solutions one can use to determine the work done in lifting a book as part of a discussion of work and *before* the introduction of potential energy. One is calculus based, the other is noncalculus based, but in both instances the solutions are obtained directly from the definition of work.

The problem is solved almost trivially using calculus. We consider it a one-dimensional problem and assume that the book suffers an instantaneous acceleration, a_y, while the work is being done; clearly, a_y is a function of time t. So, at any instant the net force on the book is

$$F = m(g + a_y), \tag{1}$$

and the incremental work done is

$$dW = Fdy = m(g + a_y)dy.$$

Therefore, the total work done in lifting the book through a height h is

$$W = \int_0^h m(g + a_y)dy = mgh + m\int_0^h a_y dy. \tag{2}$$

However, the second integral on the right-hand side is zero, since

$$\int_0^h a_y dy = \int_0^h \frac{dv}{dt}dy = \int_{v_0}^{v_f} \frac{dy}{dt}dv = \int_{v_0}^{v_f} vdv, \tag{3}$$

and the initial (v_0) and final (v_f) velocities of the book are both zero. [Instructors will recognize Eqs. (2) and (3) as simply equating work with changes in potential and kinetic energies, but note that it is not necessary to define either of these terms here.] Therefore, the work done is

$$W = mgh,$$

and it is independent of the acceleration, etc.

In noncalculus-based classes, we have to use a different approach. As an example, let us take a case where the acceleration of the book from the floor to some position y_1, i.e., for $0 < y < y_1$, is a_1 and for $y_1 < y < h$ the acceleration is a_2. At $y = y_1$, the velocity of the book is v_1. Since its initial and final velocities are zero, we have

$$v_1^2 = 0 + 2a_1 y_1 \quad \text{and} \quad 0 = v_1^2 + 2a_2 (h - y),$$

so that

$$a_1 = \frac{v_1^2}{2y_1} \qquad \text{and} \qquad a_2 = -\frac{v_1^2}{2(h - y_1)}.$$

Therefore, the total work done in lifting the book onto the table is

$$m(g + a_1)y_1 + m(g + a_2)(h - y_1) = mgh. \qquad (5)$$

Although this example may not constitute a proof, it is enough to satisfy most students that, indeed, the work done depends only on the mass and height of the table and not at all on *how* the work is done. Of course, one can go beyond what I have offered here. In the case of a time-varying acceleration, for example, one can use a graphical method for determining work, which involves summing small rectangles,[1] something one can carry out very easily numerically using a spreadsheet.

Reference

1. See for example, John D. Cutnell and Kenneth W. Johnson, *Physics,* 5th ed. (Wiley, New York, 2001), Sect. 6.9.