Energy and the Confused Student V: The Energy/Momentum Approach to Problems Involving Rotating and Deformable Systems

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Energy is a critical concept in physics problem-solving, but is often a major source of confusion for students if the presentation is not carefully crafted by the instructor or the textbook. A common approach to problems involving deformable or rotating systems that has been discussed in the literature is to employ the work-kinetic energy theorem together with a "pseudowork-kinetic energy theorem" or a "center-of-mass equation." This article discusses an alternative approach that employs neither of these equations and allows students a more global and less confusing approach to such problems. The approach is demonstrated for three sample situations from the literature.

Deformable and Rotating Objects

There has been significant discussion in the literature[1-7] about difficulties in applying a work-energy approach to solutions of problems. For problems in which forces are applied to a particle or a rigid, nonrotating object in a friction-free environment, the use of the work-kinetic energy theorem,

$$W = \Delta K, \tag{1}$$

is straightforward, with K representing the kinetic energy of the particle or object. In the definition of work, as discussed in the first article[8] in this series, the displacement is that of the point of application of the force. For a rigid, nonrotating object, which we will call from now on a *particle* because it can be modeled as such, this displacement is the same as that of the particle.

Now consider a force acting on a deformable system or one that rotates. In these types of problems, the displacement of the point of application of a force on the system may be different from the displacement of the center of mass of the system. A number of approaches have been offered for these types of problems. Many involve a formalism in which Newton's second law is integrated to arrive at

$$\int \sum \mathbf{F}_{\text{ext}} \cdot d\mathbf{r}_{\text{CM}} = \Delta\left(\tfrac{1}{2} m v_{\text{CM}}^2\right). \tag{2}$$

In this expression, the integral of the net external force on the system over the displacement of its center of mass equals the change in the kinetic energy of its center of mass. The integral on the left of Eq. (2) is called "pseudowork" by Penchina,[2] Sherwood,[3] and Mallinckrodt and Leff.[5] This quantity is called "center-of-mass work" by Mungan.[7]

Equation (2) is called the "pseudowork-kinetic energy theorem" by Penchina[2] and Sherwood.[3] It is called the "CM (center of mass) equation" by Sherwood and Bernard.[4] Chabay and Sherwood[9] have modified an earlier approach using this equation by applying an energy principle to a "point-particle system," represented by modeling a system as if all of its mass were at the center of mass. In this approach, the displacement of interest is again that of the center of mass.

Equations (1) and (2) are used together to address a number of problems in the literature, for example in articles by Sherwood[3] and Mungan.[7] It is my intent in this article to argue that neither Eq. (1) nor Eq. (2) is the best starting point for students to begin these

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types of problems, or for that matter, any type of energy problem. In particular, Eq. (2) is an "energy-like" equation that can lead to further student confusion. There is no need to introduce a new equation such as this, especially one that will confuse students. Students taught with a carefully crafted energy approach *already have the tools they need to solve complex problems*. Therefore, the approach to these problems is straightforward and should be presented as such rather than confusing the issue with extra unnecessary equations.

The Alternative to the Work-Kinetic Energy Theorem

Traditional approaches to teaching the concept of energy begin with the work-kinetic energy theorem and then proceed to expand the equation by adding terms as new situations are encountered. These additional terms include work done by nonconservative forces, potential energy, etc. These kinds of expansions of the basic work-kinetic energy theorem are *very* difficult for novice physics students to understand and perform on their own.

I find it better to take the time to present students with a *global* equation for energy at the beginning of the discussion in mechanics and then reduce the equation accordingly for a given situation, as discussed in the fourth article[10] in this series. The global equation is the conservation of energy equation (CEE):

$$\Delta K + \Delta U + \Delta E_{\text{int}} = W + Q + T_{\text{MT}} + T_{\text{MW}} + T_{\text{ER}} + T_{\text{ET}} . \quad (3)$$

It is far easier for students to identify the terms that do not belong in a well-understood general equation than it is for them to come up with new terms that must be added to a simplified equation in a traditional approach.

Students taught with the global approach to energy will not reach for the work-kinetic energy theorem when they begin a new challenging problem, but will instead use Eq. (3). In many cases, the work-kinetic energy theorem will not be appropriate to solve the problem, so the global approach makes the problem soluble.

The Alternative to the Pseudowork or Center-of-Mass Equation

Let us now turn our attention to the use of Eq. (2) to solve problems in combination with the work-kinetic energy theorem. There are the following disadvantages to this approach:

1. The integral on the left of Eq. (2) is not work because the displacement in the equation is that of the center of mass of the system, not that of the point of application of the force. By calling the left side of Eq. (2) "pseudowork" or "center-of-mass work," we are suggesting too strongly that the integral is some form of work. The instructor who has carefully identified the displacement in the definition of work as that of the point of application of the force will have difficulty with Eq. (2) in presenting students with a term that looks like work but includes a displacement that is defined differently.
2. The point is made in the literature[3] that Eq. (2) is not an energy equation because it is generated from a dynamical equation, Newton's second law. Students have difficulty buying into this because that sure looks like work on the left-hand side and that sure looks like kinetic energy on the right-hand side of Eq. (2).
3. In our teaching, we stress the importance of solving problems from fundamental principles. There is *one* fundamental principle in an energy approach: conservation of energy. There is *one* equation associated with this principle: the conservation of energy equation. In the global approach to energy, Eq. (1) is a specific reduction of the general conservation of energy equation in a special case. Because Eq. (2) looks so much like an energy equation, students are confused by the fact that they appear to be using two equations from an energy approach when only one exists.

These disadvantages disappear if a different approach is used in place of Eq. (2). It is easy to show that Eq. (2) is *mathematically equivalent to the impulse-momentum theorem*, because both are generated from Newton's second law. The impulse-momentum theorem,

$$\int \sum \mathbf{F}_{\text{ext}} dt = m\Delta \mathbf{v}_{\text{CM}}, \quad (4)$$

carries the same information as the center-of-mass equation. Students *already have* the tool of the impulse-momentum theorem in their toolbox. Why introduce yet another equation, Eq. (2), that carries the same information? Furthermore, why introduce an energy-like equation to students but tell them that it's not a true energy equation?

Therefore, in the energy/momentum approach, we use the impulse-momentum theorem for problem-solving in place of the pseudowork or center-of-mass equation. The student is already familiar with this equation, so there is no reason to introduce a new energy-like equation that confuses the understanding of work.

The Energy/Momentum Approach

In the energy/momentum approach discussed in this article, the two equations used to address these problems are the CEE, Eq. (3), and the impulse-momentum theorem, Eq. (4), rather than Eqs. (1) and (2), as in the traditional approach. The energy/momentum approach has the following advantages:

1. There is no need to introduce "pseudowork" or "center-of-mass work." There is only one type of work done on a system, the work as calculated with the standard definition.
2. Equation (4) is clearly not an energy equation so it will not be confused with other, true energy equations.
3. Problems involving deformable or rotating systems can be solved by selecting one fundamental principle from an energy approach, the CEE, and one principle from a momentum approach, the impulse-momentum theorem.

Example Problems

Using the energy/momentum approach, let us address three sample problems. The first problem below is a simple situation involving a deformable system described by Sherwood.[3]

Problem 1

Figure 1(a) shows an overhead view of the initial configuration of two pucks of mass m on a frictionless surface tied together with a string of length ℓ and negligible mass. At time t = 0, a constant force of magnitude F begins to pull to the right on the center point of the string. At time t, the moving pucks strike each other and stick together. At this time, the point of application of the force has moved through a distance d and the pucks have attained a speed v, as shown in Fig. 1(b). What is v and how much of the energy transferred into the system from the surroundings has been transformed to internal energy?

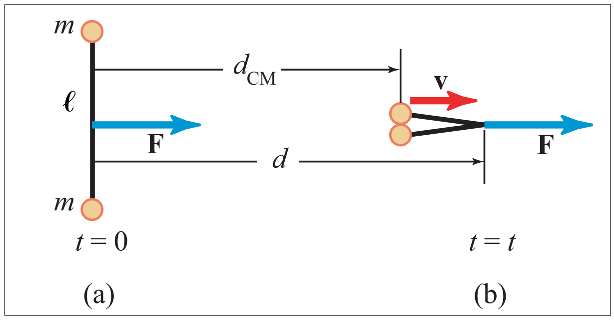

Fig. 1. Two pucks are connected by a string of length ℓ. A constant force of magnitude F pulls on the center point of the string, causing the pucks to move to the right as well as toward each other. When they collide, the collision is perfectly inelastic.

The solution described here arrives at the same result as Sherwood by using the energy/momentum approach. We identify the system as the two pucks. Because the system is deformable, the distance that the center of mass moves during this process is not the same as the distance that the point of application of the force moves, as shown in Fig. 1(b). From this figure (modeling the pucks as having zero size), we see that $d_{CM} = d - \ell/2$.

The fact that the force on the system is constant leads to a constant acceleration of the center of mass of the system during the time interval $t = 0$ to $t = t$. For constant acceleration of the center of mass starting from rest, its average velocity over the time interval of interest is half the final velocity. Therefore, the time interval for the center of mass of the system to move a distance d_{CM} from rest to a final speed v is

$$\Delta t = \frac{d_{CM}}{v_{avg}} = \frac{d - \ell/2}{v/2} = \frac{2d - \ell}{v}. \tag{5}$$

Now, the impulse-momentum theorem, Eq. (4), gives us

$$F\left(\frac{2d-\ell}{v}\right) = (2m)(v - 0). \tag{6}$$

Equation (6) can be solved for the speed:

$$v = \sqrt{\frac{F(2d-\ell)}{2m}}. \tag{7}$$

The conservation of energy equation for this system reduces to

$$\Delta K + \Delta E_{\text{int}} = W + Q + T_{\text{MW}}, \tag{8}$$

where K is the combined kinetic energy of both pucks and ΔE_{int} is the increase in internal energy of the pucks as they undergo their perfectly inelastic collision. The work W is done on the system by the force F. The term Q represents energy transfer into the surrounding air or surface from the warm pucks (after the collision) by heat, and T_{MW} represents energy transfer by sound from the pucks due to the collision. Let us assume that these last two terms are negligible[11] so that the conservation of energy equation further reduces to

$$\Delta K + \Delta E_{\text{int}} = W. \tag{9}$$

In terms of the given parameters, this equation can be written as

$$\left[\tfrac{1}{2}(2m)v^2 - 0\right] + \Delta E_{\text{int}} = Fd. \tag{10}$$

This equation can be solved for ΔE_{int} and Eq. (7) substituted for v:

$$\begin{aligned} \Delta E_{\text{int}} &= Fd - \tfrac{1}{2}(2m)v^2 \\ &= Fd - \tfrac{1}{2}(2m)\frac{F(2d-\ell)}{2m} \\ &= Fd - F\left(d - \frac{\ell}{2}\right) = \tfrac{1}{2}F\ell. \end{aligned} \tag{11}$$

The next two example problems were posed by Mungan[7] and introduced in the first article[8] in this series. Mungan solves these problems by introducing such concepts as "center-of-mass work" and "particle work." As shown below, there is no need to introduce extra types of work-like quantities: the student taught with the energy/momentum approach described above already has the tools to solve this problem.

Problem 2

Consider the physical situation shown in Fig. 2. A cylindrically symmetric spool of mass m and radius R sits at rest on a horizontal table with friction. The spool is pulled to the right with a constant horizontal force of magnitude T by a hand on a string of negligible mass wrapped around the axle of radius r. As a result, the spool rolls without slipping a distance L along the table. Find (a) the final translational speed of the spool and (b) the value of the friction force f.

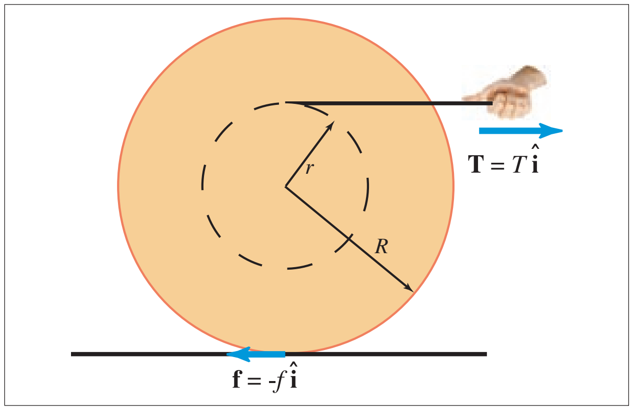

Fig. 2. A spool of radius *R* is pulled by means of a force of magnitude *T* applied to a string wrapped around an axle of radius *r*. The spool is in contact with a horizontal table that applies a friction force of magnitude *f*. The spool rolls without slipping.

Mungan solves part (a) by using Eq. (1), in which he calls the work W "particle work." This name could be confusing for students because the spool is clearly not a particle. The energy/momentum approach starts with the CEE, Eq. (3), recognizing that work is done on the system of the spool and string by the hand, resulting in only one type of energy in the system, kinetic energy. The CEE in this particular case reduces to Eq. (1). The kinetic energy of the system has two components, translational kinetic energy of the center of mass and rotational kinetic energy about the center of mass:

$$W = \Delta K = \Delta K_{\text{trans}} + \Delta K_{\text{rot}}. \tag{12}$$

If the center of mass of the spool moves through a displacement of magnitude L, the point of application of the force applied on the string by the hand moves through a displacement with magnitude $L(1 + r/R)$. Consequently, Eq. (12) gives us

$$TL\left(1 + \frac{r}{R}\right) = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega^2, \tag{13}$$

where I is the moment of inertia of the spool about its center of mass. Applying the nonslip rolling condition $\omega = v_{\text{CM}}/R$ gives

$$v_{\text{CM}} = \sqrt{\frac{2TL(1 + r/R)}{m(1+\gamma)}}, \tag{14}$$

where $\gamma \equiv I/(mR^2)$. This result agrees with Mungan's and requires only the standard conservation of energy equation used in the global approach.

For part (b), Mungan uses two equations, the work-kinetic energy theorem for the center of mass and a rotational version of the work-kinetic energy theorem. Combining these equations leads to Mungan's result:

$$f = \frac{\gamma - r/R}{1+\gamma}T. \tag{15}$$

In the energy/momentum approach, we apply one equation, the impulse-momentum theorem, to the system:

$$(T - f)\Delta t = m(v_{CM} - 0) = mv_{CM}. \tag{16}$$

The fact that the net force is constant leads to a constant acceleration of the center of mass of the spool. For constant acceleration of an object starting from rest, its average velocity is half the final velocity. Therefore, the time interval for the center of mass of the spool to move a distance L from rest to a final speed v_{CM} is

$$\Delta t = \frac{L}{v_{CM,avg}} = \frac{2L}{v_{CM}}. \tag{17}$$

As a result, Eq. (16) becomes

$$(T - f)\frac{2L}{v_{CM}} = mv_{CM}. \tag{18}$$

Solving this equation for f and substituting v_{CM} from Eq. (14) gives us Eq. (15).

Problem 3

As shown in Fig. 3, two rigid blocks, each of mass m, are at rest on a level, frictionless table. They are connected by a spring of negligible mass having force constant k. The separation distance of the blocks when the spring is relaxed is L. During a time interval Δt, a constant force of magnitude F is applied horizontally to the left block, moving it through a distance x_1. During this time interval, the right block moves to the right through a distance x_2. After the time interval Δt, the force is removed. Find (a) the resulting speed of the center of mass of the system of the spring and the two blocks, and (b) the total energy associated with vibration relative to the center of mass of the system after the force is removed.

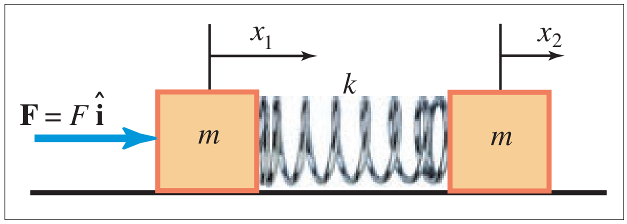

Fig. 3. Two blocks of mass *m* are connected by a spring of force constant *k*. The system sits at rest with the spring relaxed on a frictionless table. A constant force of magnitude *F* is applied to the left-hand block, moving it through a distance x_1. During this time interval, the spring causes the right-hand block to move to the right through a distance x_2.

Mungan again uses equations representing center-of-mass work and particle work to solve parts (a) and (b). While these equations allow these two parts of the problem to be solved in two lines each, I feel that the solutions come at the expense of student conceptual understanding because of the use of extra energy-like equations. To solve part (a) in the energy/momentum approach, we apply the impulse-momentum theorem to the system of the spring and the two blocks. From Eq. (4), recognizing that the force F is constant during the time interval Δt while the force is applied,

$$F\Delta t = (2m)(v_{CM} - 0) = 2mv_{CM}. \tag{19}$$

During the time interval Δt, the center of mass of the system moves a distance $\frac{1}{2}(x_1 + x_2)$ with constant acceleration. Therefore,

$$\Delta t = \frac{\frac{1}{2}(x_1 + x_2)}{v_{CM,avg}} = \frac{\frac{1}{2}(x_1 + x_2)}{\frac{1}{2}v_{CM}} = \frac{(x_1 + x_2)}{v_{CM}}. \tag{20}$$

Substituting Δt into Eq. (19) gives

$$v_{CM} = \sqrt{F\frac{(x_1 + x_2)}{2m}}. \tag{21}$$

To find the vibrational energy in part (b), we apply the CEE to the system of the spring and two blocks. We know that the kinetic energy of the system can be expressed as $K = K_{CM} + K_{vib}$, where K_{vib} is the kinetic energy of the blocks relative to the center of mass due to their vibration. Therefore, the CEE becomes

$$\Delta K_{CM} + \Delta K_{vib} + \Delta U_{vib} = W, \tag{22}$$

where U_{vib} is the elastic potential energy stored in the spring when the separation of the blocks is some value other than L. Recognizing that $K_{vib} + U_{vib} = E_{vib}$ and that the initial values of the kinetic energy of the center of mass and the vibrational energy are zero, this equation becomes

$$K_{CM} + E_{vib} = Fx_1 \,. \tag{23}$$

Note that the work done on the system is the product of the force F and the displacement of magnitude x_1 of the point of application of this force. Therefore,

$$E_{vib} = Fx_1 - K_{CM} = Fx_1 - \tfrac{1}{2}(2m)v_{CM}^2. \tag{24}$$

Substituting for v_{CM} from Eq. (21), we find

$$E_{vib} = Fx_1 - \tfrac{1}{2}(2m)\left(F\frac{(x_1 + x_2)}{2m}\right) = F\frac{(x_1 - x_2)}{2}. \tag{25}$$

It should be noted that all three problems discussed above involve constant forces, which allows the time interval Δt to be evaluated easily. The energy/momentum approach remains straightforward if the force applied to a system varies in a known way with time; the integral in Eq. (4) can be evaluated for such a force. If the force varies in a known way with position, the solution becomes more problematic. On the other hand, a problem involving a deformable or rotating system under the influence of a force that varies with position is most likely to be above the level of the introductory course taught by most physics instructors. For such a problem, if the force is a well-behaved function of position, the system can be expanded to include the agent exerting the force. The force is now internal to an isolated system and can be represented by a potential energy. Then an analysis of the isolated system can be used to find the solution.

Conclusion

Energy approaches to problems involve challenges to physics students. These challenges are made more difficult by presenting a restricted view using only the work-kinetic energy theorem and then presenting specialized versions of the theorem and extra energy-like equations. Time would be well spent during the mechanics portion of the introductory course discussing the global conservation of energy equation, and using it to analyze problems.

For problems involving deformable or rotating systems, the use of the impulse-momentum theorem rather than a pseudowork, particle work, or center-of-mass equation has distinct advantages. The use of the impulse-momentum theorem allows students to apply a principle from momentum that they have already learned in addition to the CEE from the energy approach, without the need for extra energy-like equations.

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11. Sherwood eliminates the heat and sound terms by lumping them together with the internal energy ΔE_{int}. While this approach gives the same mathematical result, I find that it suffers from the conceptual clash of combining forms of energy transfer out of the system (heat, sound) with a form of energy storage (internal energy).

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